THE JOINT EMBEDDING PROPERTY AND MAXIMAL MODELS

JOHN T. BALDWIN, MARTIN KOERWIEN, AND IOANNIS SOULDATOS

Abstract. We introduce the notion of a 'pure' Abstract Elementary Class to block trivial counterexamples. We study classes of models of bipartite graphs and show:

Main Theorem (cf. Theorem 3.5.2 and Corollary 3.5.6): If $\langle \lambda_i : i \leq \alpha < \aleph_1 \rangle$ is a strictly increasing sequence of characterizable cardinals (Definition 2.1) whose models satisfy JEP($< \lambda_0$), there is an $L_{\omega_1,\omega}$-sentence ψ whose models form a pure AEC and

(1) The models of ψ satisfy JEP($< \lambda_0$), while JEP fails for all larger cardinals and AP fails in all infinite cardinals.
(2) There exist $2^{\lambda_i^+}$ non-isomorphic maximal models of ψ in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality; and
(3) ψ has arbitrarily large models.

In particular this shows the Hanf number for JEP and the Hanf number for maximality for pure AEC with Löwenheim number $\aleph_0$ are at least $\beth_{\omega_1}$. We show that although $AP(\kappa)$ for each κ implies the full amalgamation property, $JEP(\kappa)$ for each κ does not imply the full joint embedding property.

We prove the main combinatorial device of this paper cannot be used to extend the main theorem to a complete sentence.

We investigate in this paper the spectra of joint embedding and of maximal models for an Abstract Elementary Class (AEC), in particular for AEC defined by universal $L_{\omega_1,\omega}$-sentences under substructure. Our main result provides a collection of bipartite graphs whose combinatorics allows us to construct for any given countable strictly increasing sequence of characterizable cardinals (λ_i), a sentence of $L_{\omega_1,\omega}$ whose models have joint embedding below λ_0 and $2^{\lambda_i^+}$-many maximal models in each λ_i^+, but arbitrarily large models. Two examples of such sequences (λ_i) are: (1) an enumeration of an arbitrary countable subset of the $\beth_\alpha$, $\alpha < \omega_1$, and (2) an enumeration of an arbitrary countable subset of the $\aleph_n$, $n < \omega$.

We give precise definitions and more details in Section 1. In Section 2, we describe our basic combinatorics and the main constructions are in Section 3. We now provide some background explaining several motivations for this study.

In first order logic, work from the 1950's deduces syntactic characterizations of such properties as joint embedding and amalgamation via the compactness theorem. The syntactic conditions immediately yield that if these properties hold in one cardinality they hold in all cardinalities. For AEC this situation is vastly different. In fact, one major stream studies what are sometimes called Jónsson classes that satisfy: amalgamation, joint embedding, and have arbitrarily large models. (See, for example, [**?**, **?**, **?**] and a series of paper such as [**?**].) Without this hypothesis the properties must be parameterized and the relationship between, e.g. the Joint Embedding Property (JEP) holding in various cardinals, becomes a topic for study. In [**?**] Grossberg conjectures the existence of a Hanf number for the Amalgamation Property (AP): a cardinal $\mu(\lambda)$ such that if an AEC with Löwenheim number λ has the AP in some cardinal greater than $\mu(\lambda)$ then it has the amalgamation property in all larger cardinals. Boney [**?**] makes great progress on this problem by showing that if κ is strongly compact and an AEC $\boldsymbol{K}$ is categorical in λ^+ for some $\lambda \geq \kappa$, then $\boldsymbol{K}$ has JEP and AP above κ. Baldwin and Boney, [**?**], show that if there is a strongly compact cardinal then it is an

Date: February 20, 2015.
2010 *Mathematics Subject Classification.* Primary 03C48 Secondary 03C75, 03C52, 03C30.
Key words and phrases. Abstract Elementary Class, Joint Embedding, Amalgamation, Maximal Models, Hanf Number for Joint Embedding, Characterizable Cardinals.

upper bound on the Hanf number for joint embedding. Our results here give much smaller but concrete lower bounds in ZFC for the Hanf number of JEP, again with no categoricity involved.

The lack of a syntactic condition for joint embedding or amalgamation is a symptom of the lack of a good notion of 'complete' for AEC's. In particular various 'trivial' counterexamples arise from mere (or slightly disguised) disjunction of sentences. We introduce the notion of a pure AEC to address this issue. We attempted in [**?**, **?**] to find other notions of 'complete' which might provide a more robust substitute for 'joint embedding'. Often 'complete for $L_{\omega_1,\omega}$' is taken as a good completeness notion. It certainly is a robust notion as witnessed by Shelah's categoricity theorem for such sentences. But the examples of e.g [**?**, **?**] show that even such sentences do not guarantee even joint embedding in all cardinals. Here we are considering weakenings of the full joint embedding property. But we show (Theorem 3.2.5), that (even weak versions of) Lemma 3.5.2 cannot be extended to a complete sentence for the combinatorics here.

Kolesnikov and Lambie-Hanson [**?**] study a family of AEC's called coloring classes, They show that for these classes[1] the amalgamation property is equivalent to disjoint amalgamation and the Hanf number for amalgamation is $\beth_{\omega_1}$; specific classes fail disjoint amalgamation for the first time arbitrarily close to $\beth_{\omega_1}$. Our examples have arbitrarily large models and no maximal model above $\beth_{\omega_1}$. But specific classes have maximal models arbitrarily close to $\beth_{\omega_1}$; we specify the cardinalities of the maximal models exactly.

Baldwin, Koerwien, and Laskowski [**?**] exploit excellence in appropriate classes, axiomatized by universal sentences in $L_{\omega_1,\omega}$ to construct complete sentences in $L_{\omega_1,\omega}$ that uniformly homogeneously characterize (ϕ_α has no model of cardinality $> \aleph_\alpha$) cardinals below $\aleph_\omega$. We use these examples as an input to Corollary 3.5.6(1) constructing AEC that have maximal models in a countable set of cardinals less than $\aleph_\omega$. Similarly we use Morley's example to show the Hanf number for JEP is at least $\beth_{\omega_1}$ in Corollary 3.5.6(2).

1. JEP and Pure AEC

One can trivially augment any AEC $\boldsymbol{K}$ by adding structures below the Löwenheim-Skolem number LS($\boldsymbol{K}$) which have no extensions; to avoid such trivialities the following assumption applies to the rest of the paper.

Assumption 1.1. *For each AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, we work at or above the Löwenheim-Skolem number LS($\boldsymbol{K}$).*

In this section we spell out the parameterized notions of joint embedding and introduce the notion of pure and hybrid AEC. We then show that there is no real theory possible if hybrid AEC are allowed.

Definition 1.2. *The AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the* joint embedding property *at the infinite cardinal κ (JEP(κ)) if for any two models A, B of cardinality κ have a common $\prec_{\boldsymbol{K}}$ extension C.*

If this condition holds for models A, B of any cardinality $\leq \kappa$ ($< \kappa$) we write JEP($\leq \kappa$) (JEP($< \kappa$)). In particular, $|A|$ and $|B|$ can be different.

The full-joint embedding property *(full-JEP) is the JEP with no restriction on the sizes of A, B.*

The AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the amalgamation property *at the infinite cardinal κ (AP(κ)) if for any three models A, B, C each of cardinality κ there are $\prec_{\boldsymbol{K}}$-maps of B and C into an extension D which agree on A. We use cardinal parameters as for joint embedding.*

[1]For easy comparison, we restrict their results to countable languages.

By Assumption 1.1, κ is greater or equal to $\mathrm{LS}(\boldsymbol{K})$ and without loss of generality we can assume that C in the definition of JEP and D in the definition of AP, both have size κ.

The following easy consequences of the definitions show there are some subtleties in the relation between joint embedding, maximal models and arbitrarily large model.

Lemma 1.3. *Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ be an AEC*

1. *If there are no maximal models then $\boldsymbol{K}$ has arbitrarily large models.*
2. *If $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ satisfies JEP($\leq \kappa$) and has a model in power κ then any model extends to one of size of κ; thus*
3. *If $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has full-JEP and has arbitrarily large models then $\boldsymbol{K}$ has no maximal models.*
4. *If $\boldsymbol{K}$ has two non-isomorphic maximal models in power κ, JEP(κ) fails.*

We will construct an AEC with at least two maximal models in a cardinal λ^+. Condition 4) says the most JEP possible is JEP($\leq \lambda$). Some of our examples will satisfy this. For others we settle for JEP($< \lambda$).

We show that without the hypothesis of full-JEP the implication from arbitrarily large models to no maximal models fails on various countable sets of cardinals. There are some trivial examples for this (see Corollary 1.6), where one just takes disjunctions of sentences (in disjoint vocabularies). However, the disjunction of two sentences introduces properties that are clearly artificial. In particular, one can find sentences with maximal models in any countable set of cardinals by putting them in disjoint vocabularies and taking a disjunction. Such a class does not have JEP in any cardinal. We eliminate these trivial examples using the following definition. Moreover, our examples also satisfy JEP($\leq \lambda$), for some infinite λ.

Definition 1.4.

- *Let $\boldsymbol{K}$ be a collection of τ-structures and τ_1 be a subset of τ. Then $\boldsymbol{K}_{\tau_1}$ is the subcollection of $\boldsymbol{K}$ of models where all symbols not in τ_1 have the empty interpretation.*
- *An AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ in a vocabulary τ is called* hybrid *if $\tau = \tau_1 \cup \tau_2$, $\boldsymbol{K} = \boldsymbol{K}_{\tau_1} \cup \boldsymbol{K}_{\tau_2}$ and at least one of τ_1, τ_2 is not equal to τ.*
  If $\boldsymbol{K}$ is not hybrid then it is pure.

The most trivial example of a hybrid AEC is defined by the disjunction of sentences in disjoint vocabularies. The definition allows a more subtle version where the vocabularies can overlap but one of the classes forces some of the relations to be empty. Lemma 1.5 provides an example of how hybrid AEC that are not just disjunctions of sentences in disjoint vocabularies give trivial counterexamples.

Note that trivially JEP($< \kappa$) for all κ, or AP($< \kappa$) for all κ, imply full-JEP and full-AP respectively. On the contrary, Lemma 1.5 proves that the assumption JEP($< \kappa$), for all κ, can not be replaced by the assumption JEP(κ), for all κ. In addition (see Corollary 1.6) the example in Lemma 1.5 has AP(κ), for all uncountable κ, but fails AP($\aleph_0$) and thus it fails AP($< \kappa$), for all uncountable κ. So there is a genuine distinction between these types of conditions. This is an important distinction since the definition of a good-κ frame requires the weaker AP(κ) and not AP($< \kappa$).

Lemma 1.5. *The full-Joint Embedding Property is* not *equivalent to the conjunction of JEP(κ), for all infinite κ.*

Proof. Let τ be the vocabulary $\{V, U, E, <\}$, where the V, U are unary predicate symbols, and $E, <$ are binary predicate symbols. Consider ϕ_1 to be the conjunction of τ-sentences asserting:

1. V, U partition the universe;
2. $(U, <)$ is well-ordered in order type ω; and

(3) E defines a bijection from V onto U.

Let ϕ_2 be the conjunction of

(1) U is empty, V is infinite and equals the universe; and
(2) $<, E$ are empty

Let $\boldsymbol{K}$ be the collection of models of $\phi_1 \vee \phi_2$, and let $\prec_{\boldsymbol{K}}$ be the substructure relation.

If $\mathcal{M}_1, \mathcal{M}_2$ are two models of ϕ_2 of the same cardinality κ, then they are isomorphic so can be jointly embedded. If $\mathcal{M}_1, \mathcal{M}_2$ are two (necessarily countable) models of ϕ_1, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic. If $\mathcal{M}_1$ is a model of ϕ_1 and $\mathcal{M}_2$ is a countable model of ϕ_2, then $\mathcal{M}_2$ can be embedded in $\mathcal{M}_1$. So, for all infinite κ, JEP(κ) holds.

However, if $\mathcal{M}_1$ is a countable model of ϕ_1 and $\mathcal{M}_2$ is an uncountable model of ϕ_2, $\mathcal{M}_1, \mathcal{M}_2$ have no common extension in $\boldsymbol{K}$ (as $\mathcal{M}_1$ is maximal in $\boldsymbol{K}$). So, full-JEP fails. $\square$

Corollary 1.6. *The AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ defined in Lemma 1.5 is a hybrid AEC that satisfies JEP(κ) for all infinite κ, has maximal models in $\aleph_0$, and has arbitrarily large models, but fails JEP($\leq \aleph_1$). Moreover, it satisfies AP(κ), for all uncountable κ, but it fails AP($\aleph_0$) and AP($\leq \aleph_1$).*

Question 1.7. *Is there a 'pure' example (according to Definition 1.4) to illustrate the distinction between full-JEP and JEP(κ) for all κ?*

We contrast this result with a less complicated version of a result of Shelah (Theorem 2.8 of [**?**]) which was originally stated without proof.[2].

Fact 1.8. *If an AEC has AP(κ) for every κ, then it has the (full-) Amalgamation Property.*

An easy variation of the proof shows:

Corollary 1.9. *If $\lambda < \kappa$ and an AEC satisfies JEP(λ) and AP($\leq \kappa$), then it also satisfies JEP(κ) and even JEP($\leq \kappa$).*

Thus, for the distinction made in Lemma 1.5 between JEP($\leq \aleph_1$) and the conjunction of JEP($\aleph_0$) and JEP($\aleph_1$), it is imperative for AP($\leq \aleph_1$) to fail.

We can modify the example of Lemma 1.5 to allow $(U, <)$ to be an infinite well-order of order type $\leq \kappa$ (with strong substructure as end-extension), for some cardinal κ. The resulting AEC will satisfy JEP(λ), for all infinite λ, and even JEP($\leq \kappa$), but fail JEP($\leq \kappa^+$).

Corollary 1.10. *For all infinite κ,*

(1) JEP($\leq \kappa^+$) is not *equivalent to the conjunction of JEP(λ), $\lambda \leq \kappa^+$.*
(2) JEP($\leq \kappa^+$) is not *equivalent to the conjunction of JEP($\leq \kappa$) and JEP(κ^+).*

We will see with more difficulty below that there are pure AEC which exhibit the behavior of Corollary 1.6, i.e. they have both maximal models and arbitrarily large models.

2. Basic combinatorics

In this section we set up a first-order template of bipartite graphs on sets A, B with colors from C. We introduce the requirement that there is no monochromatic $K_{2,2}$ subgraph (a complete bipartite graph on points $a_1, a_2 \in A, b_1, b_2 \in B$ with all edges the same color). Then we show restrictions on the cardinality of A and B that are imposed by restrictions on the

[2]The proof is sketched on page 134 of [**?**]. A weaker form of this result was reproved in [**?**] (Theorem 3.4), inadvertently without citation. The result also occurs in Rami Grossberg's master's thesis.

number of colors. In later sections, we will impose the restrictions on $|C|$ by characterizing them by sentences of $L_{\omega_1,\omega}$ in the following sense.

Definition 2.1. *An $L_{\omega_1,\omega}$-sentence ϕ* characterizes *an infinite cardinal κ, if ϕ has models in all cardinalities $\leq \kappa$, but no model of size κ^+. In this case we say that the cardinal κ is* characterizable.

Since the Hanf number for $L_{\omega_1,\omega}$-sentences is $\beth_{\omega_1}$, it follows that all characterizable cardinals are strictly less than $\beth_{\omega_1}$.

Notation 2.2. *Let $\tau_0 = \{A, B, C, E\}$ where A, B, C are unary predicates and E is a ternary relation. Let σ_0 be the conjunction of the following statements:*

- *A, B, C are non-empty and partition the universe.*
- *$E \subset A \times B \times C$ defines a total function from $A \times B$ into C.*

As a notation, let $F(a, b)$ the unique value c such that $E(a, b, c)$ holds. A and B should be regarded as the two sides of a bipartite graph and C as the set of edge-labels. E assigns a unique label to any pair from $A \times B$.

Let σ_1 be the conjunction of σ_0 and

($*$) *for all distinct a_1, a_2 in A and b_1, b_2 in B, the four values $F(a_i, b_j)$ ($i, j \in \{1, 2\}$) are not all identical.*

We will also refer to ($*$) as "there are no monochromatic $K_{2,2}$ subgraphs". If a_1, a_2 are two distinct elements in A and for some $b \in B$, $F(a_1, b) = F(a_2, b)$, we will say that there exists a *monochromatic path* of length 2, or *monochromatic 2-path*, on a_1, a_2.

Lemma 2.3. *In any model of σ_1, if $|A| > |C|^+$ then $|B| \leq |C|$. By symmetry, the same is true if we switch the roles of A and B.*

Proof. Toward a contradiction, assume that $|B| > |C|$. For any subset D of B and any element $a \in A$, define $S(a, D) = \{F(a, d) | d \in D\}$ (which is a subset of C).

Now given any such D of size $|C|$ and any $b \in B \setminus D$, we observe that for all but $|C|$ many elements $a \in A$, $S(a, D) \subsetneq S(a, D \cup \{b\})$. Indeed, if $S(a, D) = S(a, D \cup \{b\})$, then we have some $c \in C$ and $d_a \in D$ with $F(a, d_a) = F(a, b) = c$. If a, a' are distinct elements of A and $F(a, d_a) = F(a, b) = c = F(a', d_{a'}) = F(a', b)$, then d_a and $d_{a'}$ have to be distinct. If not, a, a', d_a, b witness a violation to ($*$). So, for every color c, the set $\{a \in A | F(a, b) = c\}$ has size at most $|D|$. Since there exist $|C|$ many colors, there are at most $|D| \cdot |C| = |C|$ many elements such that $S(a, D) = S(a, D \cup \{b\})$.

Now let $(b_i | i < |C|^+)$ be a sequence of distinct elements in $B \setminus D$ and set $D_i = D \cup \{b_j | j < i\}$. For each $i < |C|^+$, let $A_i \subset A$ be the set of elements a such that $S(a, D_i) \subsetneq S(a, D_{i+1})$. Since $|A| > |C|^+$ and all $A \setminus A_i$ have size at most $|C|$, $A^* = \bigcap_{i<|C|^+} A_i$ is non-empty (in fact its complement has size at most $|C|^+$). But for any element $a \in A^*$, $S(a, D_i)$ grows at each step $i < |C|^+$ which is impossible since $S(a, D_i) \subset C$ for all i.

□

3. Maximal models in many cardinalities

In this section we prove that one can have interesting spectra of maximal models for AEC that are *pure* (see Definition 1.4). Specifically, we construct sentences in $L_{\omega_1,\omega}$ that are not just disjunctions of complete sentences. In Section 3.2, we show limitations on getting such results for complete sentences compatible with σ_1.

In [**?**] Shelah defines a *universal class* as one that is closed under substructure, union of chains, and isomorphism. He remarks that by a result of Tarski, if the vocabulary is finite, then such a class is axiomatized by a set of universal first order sentences. This generalizes to: If the vocabulary has cardinality κ, the class is axiomatized in $L_{\kappa^+,\omega}$. For simplicity here we use only countable vocabularies and $L_{\omega_1,\omega}$-sentences.

3.1. **Maximal Models.** Throughout this section $\mathcal{M}$ is a model of σ_1 of cardinality κ. Since we discuss in this subsection only the construction of extensions of a single model we are free to assume that $C \subseteq \kappa$. We write $C^{\mathcal{M}} = C \subset \kappa$ to assert that the interpretation of the predicate C is a subset of κ and use $|C|$ when we mean cardinality.

In this section we build models $\mathcal{M}$ of σ_1 that are C-maximal in the following sense.

Definition 3.1.1. *Let $|C| = \kappa$; $\mathcal{M}$ is a C-maximal model of σ_1 if $C^{\mathcal{M}} = C$ and there is no proper extension of $\mathcal{M}$ to a model $\mathcal{M}'$ of σ_1 with $C^{\mathcal{M}'} = C$.*

In applications we require that we will expand τ_0 to a vocabulary $\tau' = \tau_0 \cup \tau_1$ and study τ'-models M such that $M \restriction \tau_0 \models \sigma_1$ and $C^M \restriction \tau_1$ belongs to an AEC $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}_0})$ for the vocabulary τ_1, and $\boldsymbol{K}_0$ has models in cardinality κ but no larger, and thus it has a maximal model in κ.

Notation 3.1.2. *If $\mathcal{M} \models \sigma_1$, $|A^{\mathcal{M}}| = \kappa$ and $|B^{\mathcal{M}}| = \lambda$ then we say that $\mathcal{M}$ is a (κ, λ)-model.*

In the following construction, we reverse the procedure of Lemma 3.1.3 and build a model from a function on cardinals.

Lemma 3.1.3. *For any κ, there is a (κ^+, κ^+) model $\mathcal{M} \models \sigma_1$ such that $C^{\mathcal{M}} = \kappa$.*

Proof. Let $A^{\mathcal{M}}$ and $B^{\mathcal{M}}$ be two copies of κ^+.

Fix a function F from $\kappa^+ \times \kappa^+$ to κ such that a)[3] for all α, $F(\alpha, \alpha) = 0$ and b) for all $\alpha \in A$, $F(\alpha, \cdot)$ is a one-to-one function when restricted to the set $\{\beta \in B | \beta \leq \alpha\}$. Symmetrically, demand that for all $\beta \in B$, $F(\cdot, \beta)$ is a one-to-one function when restricted to the set $\{\alpha \in A | \alpha \leq \beta\}$. Both conditions are possible because all initial segments have size $\kappa = |C^{\mathcal{M}}|$. Then define a graph as in Notation 2.2 using this function.

Towards contradiction, assume that there are distinct α_1, α_2 in A and β_1, β_2 in B with all four values $F(\alpha_i, \beta_j)$ $(i, j \in \{1, 2\})$ identical. Without loss of generality assume that $\max\{\alpha_1, \alpha_2, \beta_1, \beta_2\} = \alpha_1$. By the choice of F, $F(\alpha_1, \beta_1)$ must be different than $F(\alpha_1, \beta_2)$. Contradiction. $\square$

Corollary 3.1.4. *For any infinite cardinal κ, the class of all models $\mathcal{N}$ of σ_1 with $C^{\mathcal{N}} = C$ where $C = \kappa$ is fixed, contains a (κ^+, κ^+)-model $\mathcal{M}$ that is C-maximal.*

Proof. By fixing $C = \kappa$ we have an (κ^+, κ^+) model $\mathcal{M}$ by Lemma 3.1.3. But there is no extension of $\mathcal{M}$ with either A or B of cardinality $> \kappa^+$ by Lemma 2.3. The collection of extensions $\mathcal{N}$ of $\mathcal{M}$ that satisfy σ_1 with $C^{\mathcal{N}} = C$ is closed under union since σ is $\forall_1$. So some extension of $\mathcal{M}$ with cardinality κ^+ must have no extension. $\square$

We can in fact give two explicit constructions that yield nonisomorphic maximal models. The first proof uses Fodor's theorem, which we state for the sake of completeness.

Fact 3.1.5 (Fodor)**.** *If f is a regressive function on a stationary set $S \subset \kappa$, then there is a stationary set $T \subset S$ and some $\gamma < \kappa$ such that $f(\alpha) = \gamma$, for all $\alpha \in T$.*

Lemma 3.1.6. *If we modify the construction of Lemma 3.1.3 to require that*

[3] This requirement is not needed now but is used in the proof of Lemma 3.1.6.

> (†) *for all $\alpha \in A$, $\alpha \geq \kappa$, the function $F(\alpha, \cdot)$ restricted to $\{\beta \in B | \beta < \alpha\}$ is onto $C - \{0\}$,*

then we obtain a C-maximal model.

Proof. Recall for each α, $F(\alpha, \alpha) = 0$. First note that if we extend B by a new point b, then there must exist some $i \in C$ and a stationary subset S_i of A such that all the edges between $s \in S_i$ and b are colored i. Without loss of generality assume that $S_i \subset \kappa^+ \setminus \kappa$.

Now, define a function g from $\kappa^+ \setminus \kappa$ to κ^+ by

$$g(\alpha) = \text{ least } \beta < \alpha \text{ such that } F(\alpha, \beta) = i.$$

By (†), g is well-defined on all $\kappa^+ \setminus \kappa$, and by the definition, g is regressive, i.e. $g(\alpha) < \alpha$. By Fodor's Theorem we get a stationary $T_i \subset S_i$ and a γ_i such that for each $t \in T_i$, $F(t, \gamma_i) = i$. But this contradicts $(*)$ of Notation 2.2. □

Notation 3.1.7. *Consider the following condition $(\ddagger)_A$.*

> $(\ddagger)_A$ *For any pair $(a, a') \in A^2$ and for any color c, there exists some $b \in B$ such that $F(a, b) = F(a', b) = c$.*

Similarly, define $(\ddagger)_B$ by exchanging the role of a's and b's in $(\ddagger)_A$, and let:

> $(\ddagger)$ *is the conjunction of $(\ddagger)_A$ and $(\ddagger)_B$.*

Lemma 3.1.8. *If $\mathcal{M}$ is a (κ^+, κ^+)-model of $\sigma_1 \wedge (\ddagger)_A$, then $B^{\mathcal{M}}$ can not be extended, and symmetrically, $A^{\mathcal{M}}$ can not be extended from a (κ^+, κ^+)-models of $\sigma_1 \wedge (\ddagger)_B$.*

Thus, if $\mathcal{M}$ is a model of $\sigma_1 \wedge (\ddagger)$ with $C^{\mathcal{M}} = \kappa$ and $C^{\mathcal{M}} = \kappa$, then $\mathcal{M}$ is C-maximal.

Proof. Assume a model satisfies $(\ddagger)_A$ with $|C^{\mathcal{M}}| = \kappa$ and $C \subseteq \kappa$. Towards contradiction, assume we can extend B by one element, say b. Since there are κ many colors and κ^+ many elements $a \in A$ to connect to b, there will be two elements $a_1, a_2 \in A$ so that both edges $(a_1, b), (a_2, b)$ get the same color c. Then $(\ddagger)_A$ gives a contradiction to $(*)$. □

Lemma 3.1.9. *There exists a (κ^+, κ^+)-model of σ_1 that satisfies $(\ddagger)$ and $C = \kappa$.*

Proof. Proceed as in the proof of Lemma 3.1.3. At every stage α choose either a pair $a_1, a_2 < \alpha$ in A or a pair $b_1, b_2 < \alpha$ in B, and some color $c \in C$. Organize the construction so that every combination of a pair and a color appears at exactly one stage. This is possible, since there are κ^+ stages and κ^+ such combinations.

Assume (a_1, a_2) and c are chosen at stage α. If there is a 2-path on (a_1, a_2) colored by c, then do nothing more than what the proof of Lemma 3.1.3 requires. If there is no such pair, require that the new edges (a_1, α) and (a_2, α) are both colored by c. This is a small violation of the requirement that $F(\cdot, \alpha)$ is 1-1; demand that this is the only violation. Make the analogous choice when (b_1, b_2) and c are chosen.

We claim that the resulting construction satisfies $(*)$ and obviously satisfies $(\ddagger)$. Towards contradiction, assume that there are distinct α_1, α_2 in A and β_1, β_2 in B with all four values $F(\alpha_i, \beta_j)$ $(i, j \in \{1, 2\})$ equal to the same value c. Without loss of generality assume that $\max\{\alpha_1, \alpha_2, \beta_1, \beta_2\} = \alpha_1$. Observe that $F(\alpha_1, \beta_1) = F(\alpha_1, \beta_2) = c$ is possible only if the pair β_1, β_2 and the color c were chosen at stage α_1. Split into two cases:

Case 1. $\alpha_2 > \beta_1, \beta_2$. Then the same observation (for α_2) proves that the same pair β_1, β_2 and the same color c that were chosen at stage α_1 were also chosen at stage α_2. But it is impossible for the same combination of pair and color to appear more than once. Contradiction.

Case 2. $\alpha_2 \leq \max\{\beta_1, \beta_2\}$. Then at stage α_1, there already exists a 2-path on (β_1, β_2) colored by c. The construction requires in this case that $F(\alpha_1, \beta_1)$ be different than $F(\alpha_1, \beta_2)$ which again yields a contradiction. So, $(*)$ holds. $\square$

The requirements of Lemma 3.1.6 and Lemma 3.1.8 are contradictory, so there are two nonisomorphic C-maximal models of σ_1.

Corollary 3.1.10. *For all infinite cardinals κ, there is a model $\mathcal{M}$ with $C^{\mathcal{M}} = \kappa$ that has two non-isomorphic extensions that are C-maximal.*

We can vary the constructions and get still other maximal models; these construction will used in the next section.

Corollary 3.1.11. *If $A^{\mathcal{M}} = \kappa^+$, A_0 is a club in κ^+ with $A_0 \cap \kappa = \emptyset$, and $C^{\mathcal{M}} \subset \kappa$ then condition $(\dagger)$ in Lemma 3.1.6 can be relaxed to the following condition.*

> $(\dagger)_{A_0}$ *For all $\alpha \in A_0$, the function $F(\alpha, \cdot)$ restricted to the set $\{\beta \in B | \beta < \alpha\}$ is onto $C - \{0\}$.*

and we still get that $\mathcal{M}$ is C-maximal.

Corollary 3.1.12. *If $A^{\mathcal{M}} = \kappa^+$, A_0 is a subset of A of size κ^+, and $C^{\mathcal{M}} \subset \kappa$, then $(\ddagger)_A$ can be relaxed to*

> $(\ddagger)_{A_0}$ *For any pair (a, a'), $a, a' \in A_0$, and for any color c, there exists some $b \in B$ such that $F(a, b) = F(a', b) = c$.*

and we still get that $\mathcal{M}$ is C-maximal.

Notice that while condition $(\ddagger)$ can be expressed by a first-order sentence in the same vocabulary as σ_1, this is not the case for $(\dagger)$ and $(\dagger)_{A_0}$. The latter conditions make use of the ordering $<$ that we used during the proof which is not part of the vocabulary.

Corollary 3.1.12 will be used to construct infinitely many nonisomorphic maximal models of σ_1 in Section 3.4. The existence of maximal models is complemented by the following lemma.

Lemma 3.1.13. *For any κ, there is a model $M \models \sigma_1$ with $|A|$ arbitrary large, $|B| \leq \kappa$ and $|C| = \kappa$.*

Proof. Let A be an arbitrary set, $B = \{b_\alpha | \alpha < \gamma \leq \kappa\}$, and $C = \{c_\alpha | \alpha < \kappa\}$ such that A, B, C are pairwise disjoint. For any $a \in A$ and $b_\alpha \in B$, set $F(a, b_\alpha) = c_\alpha$. We cannot have a contradiction to $(*)$ since each element in B is connected only by edges of a fixed color and distinct elements in B get distinct colors. $\square$

Corollary 3.1.14. *For any infinite cardinal κ, the class of all models of σ_1 with $|C| = \kappa$ has arbitrary large models. Moreover, in any model larger than κ^+, exactly one of A or B has to be no larger than κ.*

3.2. **Failure for Complete Sentences.** We show that our main combinatorial idea does not support the maximal model spectra given above, if the $L_{\omega_1,\omega}$-sentence is required to be complete. For this we need to formalize the consequences of our two types of constructions of maximal models. The next lemma proves that the models of σ_1 given in Lemma 3.1.13 are typical of (λ, κ)-models, where $\lambda \geq \kappa^+$. We need one definition first.

Definition 3.2.1. *Let $\mathcal{M} = (A, B, C, E)$ be colored by F. For $a \in A$, let $C_a = range(F(a, \cdot))$, and for $c \in C_a$ let*

$$B_{a,c} = \{b \in B | F(a, b) = c\}.$$

Lemma 3.2.2. *Let $\mathcal{M}$ be a (λ,κ)-model of σ_1, $\lambda \geq \kappa^+$, such that $|C^{\mathcal{M}}| = \kappa$. For all but κ many $a \in A$ and for all $c \in C_a$, $|B_{a,c}| = 1$.*

Proof. Assume otherwise, i.e. there are at least κ^+ many $a \in A$ such that there exists some $c_a \in C_a$ so that $|B_{a,c_a}| \geq 2$. Call A_0 the set of these a's. Since A_0 has size κ^+ and C has size κ, we can restrict A_0 to some subset A_1 of size κ^+ such that $c_a = c$, for all $a \in A_1$. Then, for each $a \in A_1$ choose a 2 element subset $B'_{a,c}$ of $B_{a,c}$. Since there are only κ many 2-element subsets of κ, there exist $a_1, a_2 \in A_1$, $B'_{a_1,c} = B'_{a_2,c}$. But a_1, a_2 witness that $(*)$ is violated. Contradiction. □

Now we formalize this distinction.

Lemma 3.2.3. *Let τ_1 be the (first-order) statement: "There exists some $a \in A$ so that for all $c \in C_a$, $|B_{a,c}| = 1$". If $|C| = \kappa$ and $\lambda \geq \kappa^+$, then any (λ,κ)-model of σ_1 satisfies τ_1, while τ_1 is obviously false in (κ^+,κ^+)-models.*

Corollary 3.2.4. *There is no model $\mathcal{M}$ of size κ^{++} such that $|C^{\mathcal{M}}| = \kappa$ and $\mathcal{M}$ satisfies the conjunction $\sigma_1 \wedge \neg\tau_1$.*

Now we show this combinatorics will not give a complete sentence with two maximal models.

Theorem 3.2.5. *For each κ and each $\tau' \subseteq \tau_0$, there is no complete τ'-sentence ϕ_κ such that (a) ϕ_κ allows at most κ colors, (b) ϕ_κ is consistent with σ_1, (c) ϕ_κ has maximal models in some cardinal $\lambda \geq \kappa^+$ and (d) ϕ_κ has arbitrarily large models.*

Proof. By Lemma 2.3, if ϕ_κ has a model of cardinality $\lambda \geq \kappa^{++}$, this is a (λ,κ)-model. Then by Corollary 3.2.3, ϕ_κ is consistent with τ_1. In particular, ϕ_κ does not have any (κ^+,κ^+)-models. So, all models of ϕ_κ of size $\lambda \geq \kappa^+$ are (λ,κ)-models and by Corollary 3.2.6, any such model can not be maximal. □

We see that all sufficiently large models are extendible.

Corollary 3.2.6. *If $\mathcal{M}$ is a (λ,κ)-model of σ_1, $\lambda \geq \kappa^+$, and $|C^{\mathcal{M}}| = \kappa$, then the A-side of $\mathcal{M}$ is extendible, while keeping the B-side of $\mathcal{M}$ and C the same.*

In particular, $\mathcal{M}$ is not maximal.

Proof. By Corollary 3.2.3, τ_1 holds and let a be an element that witnesses τ_1. Extend A by adding a new element a' and letting $F(a',b) = F(a,b)$, for all $b \in B$. It is immediate that $(*)$ holds in the new model. □

Observation 3.2.7. *Before we move to the next section note that the requirement that the requirement of C-maximality that appears in the results of this section can be replaced by the requirement that C is the universe of a maximal model of an $L_{\omega_1,\omega}(\tau')$-sentence ϕ for some τ' disjoint from τ_0 and ϕ characterizes κ in the sense of Definition 2.1. More generally, we can require that C belongs to an AEC with models in cardinality κ, but no larger.*

3.3. **The Maximal Model Functor and JEP.** In this section we define a functor which takes us from an AEC which has models of size κ but no models in κ^+, to an AEC with a maximal model in κ^+ but arbitrarily large models.

For the rest of the paper $\hat{\boldsymbol{K}}_0$ is taken to depend on $\boldsymbol{K}_0$ as in the next definition. We build the construction using σ_1 from Notation 2.2.

Definition 3.3.1. *Let $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}_0})$ be an AEC. The vocabulary of $\hat{\boldsymbol{K}}_0$ is $\hat{\tau}_0 = \tau_0 \cup \tau_{K_0}$. Let $\hat{\boldsymbol{K}}_0$ be the collection of models of σ_1 with the color sort C the domain of a model in $\boldsymbol{K}_0$. Define for $M, N \in \hat{\boldsymbol{K}}_0$, $M \prec_{\hat{\boldsymbol{K}}_0} N$, if $M \subset N$ and $C^M \prec_{\boldsymbol{K}_0} C^N$.*

Lemma 3.3.2. *$(\hat{\boldsymbol{K}}_0, \prec_{\hat{\boldsymbol{K}}_0})$ is an AEC with the same Löwenheim-Skolem number as $\boldsymbol{K}_0$.*

Proof. Since σ_1 is a $\forall_1^0$-first-order sentence, $\hat{\boldsymbol{K}}_0$ is closed under direct limits. The coherence axiom is straightforward. So the only issue is to check the Löwenheim-Skolem number. Let M be a model in $\hat{\boldsymbol{K}}_0$ and X be a subset of M. Find some $C_1 \in \boldsymbol{K}_0$ such that $X \cap C^M \subset C_1$, and $|C_1| = |X \cap C^M| + LS(\boldsymbol{K}_0)$. Let $M_0 = X \cup C_1$. In particular, $C^{M_0} = C_1$ and M_0 belongs to $\hat{\boldsymbol{K}}_0$. Indeed, M_0 satisfies σ_1, since any violations of $(*)$ in M_0 would be violations of $(*)$ in M too. Contradiction. Considering that $|M_0| \leq |X| + |C_1| \leq |X| + |X| + LS(\boldsymbol{K}_0) = |X| + LS(\boldsymbol{K}_0)$, it follows that $LS(\hat{\boldsymbol{K}}_0) = LS(\boldsymbol{K}_0)$. □

Theorem 3.3.3. *Let κ be an uncountable cardinal, $\boldsymbol{K}_0$ and $\hat{\boldsymbol{K}}_0$ be as in Definition 3.3.1, and suppose $\boldsymbol{K}_0$ has models in cardinality κ, but no larger.*

Then $\hat{\boldsymbol{K}}_0$ is an AEC that satisfies the following

1. *If $\lambda \leq \kappa$ then $\boldsymbol{K}_0$ satisfies JEP($\leq \lambda$) if and only if $\hat{\boldsymbol{K}}_0$ satisfies JEP($\leq \lambda$). The equivalence extends to JEP($< \lambda$) and JEP(λ).*
2. *AP fails in all infinite cardinals;*
3. *$\hat{\boldsymbol{K}}_0$ has at least 2 maximal models in κ^+ and none in any $\lambda \neq \kappa^+$; moreover, $\hat{\boldsymbol{K}}_0$ fails JEP($\leq \lambda$), even JEP(λ), for $\lambda \geq \kappa^+$.*
4. *$\hat{\boldsymbol{K}}_0$ has arbitrarily large models; and*
5. *$LS(\hat{\boldsymbol{K}}_0) = LS(\boldsymbol{K}_0)$.*

Moreover, $\hat{\boldsymbol{K}}_0$ is a pure AEC, in the sense of Definition 1.4 if and only if $\boldsymbol{K}_0$ is pure.

Proof. First observe that since $\boldsymbol{K}_0$ characterizes κ it must contain some maximal models in κ.

(1) Clearly if $\hat{\boldsymbol{K}}_0$ satisfies JEP($\leq \lambda$) then $\boldsymbol{K}_0$ satisfies JEP($\leq \lambda$). For the converse, fix $\lambda \leq \kappa$ and suppose $\boldsymbol{K}_0$ satisfies JEP($\leq \lambda$); we show $\hat{\boldsymbol{K}}_0$ satisfies JEP($\leq \lambda$). The other two cases (JEP($< \lambda$), JEP(λ)) are similar. Let $\mathcal{M}_1 = (A_1, B_1, C_1)$, $\mathcal{M}_2 = (A_2, B_2, C_2)$ be two models in $\hat{\boldsymbol{K}}_0$ such that both M_1, M_2 have size $\leq \lambda$. Use JEP on $\boldsymbol{K}_0$ and Lemma 1.3.2 to find a common extension C of both C_1, C_2 with cardinality at most λ. Then consider the structure $(A_1 \cup A_2, B_1 \cup B_2, C)$. By identifying C_1 and C_2 with subsets of C, we can consider all existing edges as C-colored. Then assign colors to new edges in a one-to-one way. This is possible, since that there are no more than λ many edges and λ many colors. Towards contradiction assume there is a violation of $(*)$ witnessed by the edges (l, l', r, r'). If there were three old edges among these then all four vertices would be in $\mathcal{M}_1$ or $\mathcal{M}_2$. So there are two new edges, but the new edges were colored, so there can not be two new edges among (l,l',r,r') with the same color. Contradiction.

(2) For any λ, let $\mathcal{M}_i = (A_i, B_i, C_i)$, $i = 1, 2, 3$, be three models of $\hat{\boldsymbol{K}}_0$ with cardinality λ such that $\mathcal{M}_1 \subset \mathcal{M}_2, \mathcal{M}_3$ and there exist distinct $a_0, a_1, a_2 \in A_1$, distinct $c, c', c'' \in C_1$, $b_2 \in B_2$, and $b_3 \in B_3$ such that $F(a_k, b_l) = c$, for $k = 0, 1$ and $l = 2, 3$, $F(a_2, b_2) = c'$, and $F(a_2, b_3) = c''$. (Extensions $\mathcal{M}_2$ and $\mathcal{M}_3$ of any $\mathcal{M}_1$ must exist). Thus, in the disjoint amalgam of $\mathcal{M}_2$ and $\mathcal{M}_3$, a_0, a_1, b_2, b_3 witness a violation to $(*)$. But in any amalgam of M_1, M_2, M_3, the images of b_2 and b_3 must be distinct, and thus, a_0, a_1, b_2, b_3 witness a violation to $(*)$ in any amalgam.

(3) By Corollary 3.1.10 there exist two maximal models in κ^+; by Corollaries 3.1.14 and 3.2.6 no model of size $> \kappa^+$ can be maximal. By (1) no model of cardinality $\leq \kappa$ can be maximal. Since there are two maximal models in κ^+, $\hat{\boldsymbol{K}}_0$ fails JEP(κ^+) and JEP($\leq \lambda$) for $\lambda \geq \kappa^+$. To see that $\hat{\boldsymbol{K}}_0$ fails JEP(λ) for $\lambda > \kappa^+$, consider a model $\mathcal{M}_0$ of type (λ, κ) and a model $\mathcal{M}_1$ of type (λ, κ). By Corollary 3.1.14, $\mathcal{M}_0$ and $\mathcal{M}_1$ can not be jointly embedded into a model in $\hat{\boldsymbol{K}}_0$.

(4) is established by Corollary 3.1.14. The proof of (5) is from Lemma 3.3.2 □

Suppose $\boldsymbol{K}_0$ has models in cardinality κ, but no larger, and $\boldsymbol{K}_0$ satisfies JEP($\leq \kappa$). It follows from Theorem 3.3.3 that $\hat{\boldsymbol{K}}_0$ will satisfy JEP($\leq \kappa$) and have a maximal model in κ^+. This condition on $\boldsymbol{K}_0$ is very strong: there is a unique maximal model in κ. However, examples of this sort (e.g. the well-orderings of order type at most ω_1 under end-extension) are well-known.

Question 3.3.4. *Is there a complete sentence of $L_{\omega_1,\omega}$? that has more than one maximal model?*

3.4. **$\mathbf{2}^{\kappa^+}$ Nonisomorphic Maximal Models.** In this section we prove that the AEC given by Theorem 3.3.3 actually has 2^{κ^+} many nonisomorphic maximal models in κ^+. We will build a family of models of σ_1, each one starting with sets A, B, C, the first two ordered as κ^+, C ordered as κ, and with a subset C_0 of C that also has order type κ, and with $C \setminus C_0$ has cardinality κ.

We are building models of the AEC $\hat{\boldsymbol{K}}_0$ with vocabulary $\hat{\tau}_0$ using an input AEC $\boldsymbol{K}_0$ with vocabulary τ_0 to control the cardinality of the color sort. The key step in the construction is to add new relations to the vocabulary $\hat{\boldsymbol{K}}_0$ and use them to construct many models (in the expanded vocabulary). But then, we show these relations are definable in $L_{\kappa,\omega}(\hat{\tau}_0)$ and deduce many $\hat{\tau}_0$-models.

The proof goes in two steps. At the first step we "code" a linear order of order type κ on C_0. At the second step we make use of this linear order to "code" κ^+ many subsets of κ into A. By varying the construction we get 2^{κ^+} many nonisomorphic maximal models.

Recall that there exists a monochromatic 2-path (based) on some $a_1, a_2 \in A$, if there exists some $b \in B$, such that both edges (a_1, b) and (a_2, b) have the same color.

Step I Code order:

Let C be the set of colors and assume $C = \kappa$. Extend the vocabulary $\hat{\tau}_0$ to $\hat{\tau}_1$ by including a unary symbol C_0 and a binary symbol $<$. C_0 will be a subset of C and $<$ will be a linear order on C_0 of order type κ. The goal is to build a model as in Lemma 3.1.9, but this time certain 2-paths are disallowed. In particular, for all $\alpha < \kappa$ there exist two elements $l_1^\alpha, l_2^\alpha \in A$ and the 2-paths on $l_1^\alpha, l_2^\alpha \in A$ can not use any of the colors $\{\beta | \beta \leq \alpha\}$. Any other color is allowed. The resulting model is maximal, as seen by Corollary 3.4.2.

Lemma 3.4.1. *There is a $\hat{\tau}_1$-model $\mathcal{M}$ that satisfies all the following conditions.*

(0) $C^{\mathcal{M}} \restriction \tau_{\boldsymbol{K}_0} \in \boldsymbol{K}_0$.
(1) $\mathcal{M} \restriction \tau_0$ *is a* (κ^+, κ^+)*-model of* σ_1 *and* $C = \kappa$.
(2) C_0 *is a subset of* C *such that* $|C_0| = |C \setminus C_0| = \kappa$ *and* $<$ *is an order on* C_0 *of order type* κ*. We may refer to the elements of* C_0 *using ordinals* $< \kappa$.
(3) $<$ *is void outside* C_0.
(4) For every $\alpha \in C_0$*, there exist two elements* $l_1^\alpha, l_2^\alpha \in A$ *such that there exists a 2-path on* l_1^α, l_2^α *colored by* c *if and only if* $c > \alpha$ *or* $c \in C \setminus C_0$.
(5) For distinct $\alpha, \alpha' \in C_0$*, the elements* $l_1^\alpha, l_2^\alpha, l_1^{\alpha'}, l_2^{\alpha'}$ *are all distinct.*
(6) For every pair (a_1, a_2) *in* A *and for all* $c \in C$*, there exists a 2-path on* a_1, a_2 *colored by* c*, unless it is forbidden by clause* (4).

Proof. We now construct the model. Let A, B, C, C_0 be as in the first paragraph of Section 3.4 and order C_0 by $<$ so that the requirements of clauses (2) and (3) are met. For every $\alpha < \kappa$, select two elements $l_1^\alpha, l_2^\alpha \in \kappa$ so that $\alpha \neq \alpha'$ implies $\{l_1^\alpha, l_2^\alpha\} \cap \{l_1^{\alpha'}, l_2^{\alpha'}\} = \emptyset$. The rest of the proof is similar to the proof of Lemma 3.1.9, the only difference is that for

every $\alpha < \kappa$, the pair l_1^α, l_2^α given by clause (4) do not have a 2-path with any color $c \leq \alpha$. The rest of the argument remains the same. □

A priori, the conditions in Lemma 3.4.1 are not $\hat{\tau}_0$-invariant. We show in Lemma 3.4.3 that C_0 and $<$ are definable in $L_{\kappa,\omega}(\hat{\tau}_0)$ so they are.

Corollary 3.4.2. *The models that satisfy the requirements of Lemma 3.4.1 are C-maximal.*

Proof. Since the set of all l_1^α, l_2^α has size κ, the result follows from Corollary 3.1.12. □

Lemma 3.4.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two $\hat{\tau}_1$-models that satisfy the conditions of Lemma 3.4.1 and let $\mathcal{M}_1|_{\hat{\tau}_0}, \mathcal{M}_2|_{\hat{\tau}_0}$ be their reducts to vocabulary $\hat{\tau}_0$. Then any isomorphism i between $\mathcal{M}_1|_{\hat{\tau}_0}$ and $\mathcal{M}_2|_{\hat{\tau}_0}$ is also an isomorphism of $\mathcal{M}_1, \mathcal{M}_2$ (as $\hat{\tau}_1$-structures).*

We will refer to this property as "the $\hat{\tau}_0$-isomorphisms respect $C_0, <$".

Proof. We claim that both $C_0, <$ are definable in the original structure $\mathcal{M}$ by a sentence of an appropriate infinitary language in vocabulary $\hat{\tau}_0$, and therefore, preserved by $\hat{\tau}_0$-isomorphisms.

First, $C_0(x)$ is defined by "there exists a pair $(l_1^\alpha, l_2^\alpha) \in A$ such that there is no 2-path on l_1^α, l_2^α colored by x".

Second, for each ordinal $\alpha < \kappa$, let $\alpha^{\mathcal{M}}$ denote the α^{th} element of the order $<^{\mathcal{M}}$. Since $<$ has order type κ, the specification makes sense. We prove by induction on $\alpha < \kappa$ that $\alpha^{\mathcal{M}}$ is defined by a formula $\phi_\alpha(x)$ in $L_{\kappa,\omega}$ in the vocabulary $\hat{\tau}_0$.

$\phi_0(x)$: There exist two points a, a' with no 2-path colored x. But for every other color $c \neq x$, there is a c-colored 2-path on a, a'.

$\phi_\alpha(x)$: There exist two points a, a' with no 2-path colored by x or by any color y satisfying $\bigvee_{\beta<\alpha} \phi_\beta(y)$. But for every other color $c \neq x$ and $\bigwedge_{\beta<\alpha} \neg\phi_\beta(c)$, there is a c-colored 2-path on a, a'.

Now $<$ is defined by a $L_{\kappa,\omega}$-formula in the vocabulary $\hat{\tau}_0$.

$$x < y \text{ if and only if } \bigvee_{\alpha<\beta<\kappa} \phi_\alpha(x) \wedge \phi_\beta(y).$$

Since each $\phi_\alpha(x)$ is a formula in vocabulary $\hat{\tau}_0$, this proves the result. □

It also follows by a similar argument that the elements l_1^α, l_2^α are definable by a formula in $L_{\kappa,\omega}$ in the vocabulary $\hat{\tau}_0$. Consider the formula $\phi(x, y)$: "there exists a 2-path on x, y colored by c if and only if $\neg \bigvee_{\beta\leq\alpha} \phi_\beta(c)$. ". By clauses (4) and (6) of Lemma 3.4.1, $\phi(x, y)$ holds if and only if $\{x, y\} = \{l_1^\alpha, l_2^\alpha\}$. So, any $\hat{\tau}_0$-isomorphism must preserve the two-element subset $\{l_1^\alpha, l_2^\alpha\}$, for all $\alpha < \kappa$.

Step II Code subsets:

Recall that $\hat{\tau}_1 = \hat{\tau}_0 \cup \{C_0, <\}$ and extend $\hat{\tau}_1$ to $\hat{\tau}_2$ by including a new binary symbol S. S will be a binary relation that codes subsets $A_0 = \{m_\alpha | \alpha < \kappa^+\}$ of A by elements of C_0.

We also require that the set $\{l_i^\alpha | \alpha < \kappa, i = 1, 2\}$ from Step I and the set A_0 from Step II are disjoint. Using S we can assign to each $m_\alpha \in A_0$ a distinct subset S_α of C_0. The goal is to build a model that satisfies all the restrictions from Step I, plus more 2-paths are forbidden. In particular, for each $\alpha < \kappa^+$ the 2-paths based on m_0, m_α can not use any of the colors in S_α. Every other color is allowed. Once again, the resulting model is maximal (see Corollary 3.4.5).

We again add predicates, this time to code models, and then prove they are $L_{\kappa,\omega}(\hat{\tau}_0)$ definable.

Lemma 3.4.4. *There is an $\hat{\tau}_2$-model $\mathcal{N}$ that satisfies all the following conditions.*

(1) Clauses $(1)-(5)$ from Lemma 3.4.1 hold.
(2) There is a set $A_0 = \{m_\alpha : \alpha < \kappa^+\} \subset A$ such that $|A \setminus A_0| = \kappa^+$ and A_0 is disjoint from $\{l_i^\alpha | \alpha < \kappa, i = 1, 2\}$.
(3) $S(x, y)$ is a binary relation on $A_0 \times C_0$. Denote the set $\{y \in C_0 | S(m_\alpha, y)\}$ by S_α.
(4) The S_α's are distinct subsets of C_0. For all α, $|S_\alpha| = |C_0 \setminus S_\alpha| = \kappa$, and 0 does not belong to any S_α.
(5) For all $0 < \alpha < \kappa^+$, there exists a 2-path on m_0, m_α colored by c if and only if $c \in S_\alpha$.
(6) For all $a_1, a_2 \in A$ and for all c, there exists a 2-path on a_1, a_2 colored by c, unless it is forbidden by clause (5) of this Lemma or by clause (4) of Lemma3.4.1.

Proof. The proof is similar to the proof of Lemma 3.4.1 and is left to the reader. □

Corollary 3.4.5. *The models that satisfy the requirements of Lemma 3.4.4 are C-maximal.*

Proof. Since $|A \setminus (A_0 \cup \{l_i^\alpha | \alpha < \kappa, i = 1, 2\})| = \kappa^+$, the result follows from Corollary 3.1.12. □

Lemma 3.4.6. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two $\hat{\tau}_2$-models that satisfy the conditions of Lemma 3.4.4 and let $\mathcal{N}_1|_{\hat{\tau}_0}, \mathcal{N}_2|_{\hat{\tau}_0}$ be their reducts to vocabulary $\hat{\tau}_0$. Then any isomorphism i between $\mathcal{N}_1|_{\hat{\tau}_0}$ and $\mathcal{N}_2|_{\hat{\tau}_0}$ is also an isomorphism of $\mathcal{N}_1, \mathcal{N}_2$ (as $\hat{\tau}_2$-structures).*

Proof. From Step I we know that C_0 and $<$ are definable by $L_{\kappa,\omega}(\hat{\tau}_0)$-formulas. We prove that the same is true for the set $A_0 = \{m_\alpha | \alpha < \kappa^+\}$ and the sets S_α, $\alpha < \kappa^+$. The element m_0 is defined by the following $L_{\kappa,\omega}(\hat{\tau}_0)$-formula $\psi_0(x)$.

$\psi_0(x)$: there exist two distinct elements $m_1, m_2 \in A$ and two distinct colors $c_1, c_2 \in C_0$ and there is no 2-path based on x, m_1 colored by c_1, and there is no 2-path based on x, m_2 colored by c_2.

We now show the set $\{m_\alpha | 0 < \alpha < \kappa^+\}$ is defined by the formula $\psi_1(x)$.

$\psi_1(x)$: there exists some $y \neq x$ such that $\psi_0(y)$, i.e. y equals m_0, and there exists a color $c \in C_0$ and there is no 2-path based on y, x colored by c.

Then $\psi_1(x)$ holds if and only if x belongs to $\{m_\alpha | 0 < \alpha < \kappa^+\}$. Note that ψ_1 defines the whole set $\{m_\alpha | 0 < \alpha < \kappa^+\}$, but not the order of the m_α's in this set. Nevertheless, for every $\alpha < \kappa^+$, the set S_α is definable by the following formula ψ_2 which uses m_α as a parameter; ψ_2 is a reformulation of clause (5) from Lemma 3.4.4.

$\psi_2(x, m_\alpha)$: there exists some y such that $\psi_0(y)$ and there exists a 2-path on y, m_α colored by x.

Then $\psi_2(x, m_\alpha)$ holds if and only if $x \in S_\alpha$.

Since all these sentences are in vocabulary $\hat{\tau}_0$, this finishes the proof. □

Now fix Y to be some subset of κ^+ and vary the construction of Lemma 3.4.4 so that for each $0 < \alpha < \kappa^+$, $0 \in S_\alpha$ if and only if $\alpha \in Y$. Call the corresponding $\hat{\tau}_2$-structure $\mathcal{N}_Y$. If Y_1, Y_2 are two distinct subsets, then $\mathcal{N}_{Y_1}$ and $\mathcal{N}_{Y_2}$ are easily seen to be nonisomorphic as $\hat{\tau}_2$-structures. By Lemma 3.4.6, their $\hat{\tau}_0$-reducts are also nonisomorphic, which proves the following.

Theorem 3.4.7. *If $\boldsymbol{K}_0$ is an AEC that has models in cardinality κ but no larger, then $\hat{\boldsymbol{K}}_0$ from Theorem 3.3.3 has 2^{κ^+}-many nonisomorphic maximal models of type (κ^+, κ^+).*

In the next section we give three applications of Theorem 3.4.7.

3.5. **Maximal Models in Many Cardinalities.** If $\kappa < \lambda < \beth_{\omega_1}$ are two characterizable cardinals (Definition 2.1) and $\boldsymbol{K}_\kappa$, $\boldsymbol{K}_\lambda$ the corresponding AEC (in disjoint vocabularies), then the union[4] with an AEC with arbitrarily large models is an AEC (with strong substructure being in the appropriate vocabulary) with maximal models in κ and λ and arbitrarily large models. However, the union is a hybrid AEC which fails JEP in all cardinals.

If $< \lambda_i | i \leq \alpha < \aleph_1 >$ is a strictly increasing sequence of characterizable cardinals (Definition 2.1), we provide an example of a pure (Definition 1.4) AEC with maximal models in cardinalities $< \lambda_i^+ | i \leq \alpha < \aleph_1 >$, arbitrarily large models, and JEP($< \lambda_0$) holds.

For any triple (A, B, C) there is a first order sentence σ_1 asserting that A, B form a bipartite graph with C many colors that contains no monochromatic $K_{2,2}$ subgraph (see property $(*)$). The structures constructed below will contain many substructures satisfying this requirement. Rather than cluttering the paper with a careful description of the formal sentence (with different ternary relations for each colored graph) we will just assert where σ_1 holds.

We begin with the case of two cardinals.

Lemma 3.5.1. *Let $\kappa < \lambda$ and let $(\boldsymbol{K}_0^k, \prec_k)$ be an AEC in vocabulary τ^k with models in κ but no higher, and let $(\boldsymbol{K}_0^\ell, \prec_\ell)$ be an AEC in vocabulary τ^ℓ with models in λ but no higher. If both $(\boldsymbol{K}_0^k, \prec_k)$ and $(\boldsymbol{K}_0^\ell, \prec_\ell)$ satisfy JEP($< \kappa$), then there is an AEC $(\boldsymbol{K}^*, \prec_{\boldsymbol{K}^*})$ which*

(1) satisfies JEP($< \kappa$);
(2) fails AP in all infinite cardinals;
(3) has 2^{κ^+} non-isomorphic maximal models in κ^+, 2^{λ^+} non-isomorphic maximal models in λ^+, but no maximal models in any other cardinality, while JEP fails in all $\lambda \geq \kappa$;
(4) has arbitrarily large models; and
(5) $LS(\boldsymbol{K}^*) = \max\{LS(\boldsymbol{K}_0^k), LS(\boldsymbol{K}_0^\ell)\}$.

If both $(\boldsymbol{K}_0^k, \prec_k)$ and $(\boldsymbol{K}_0^\ell, \prec_\ell)$ are pure, then $(\boldsymbol{K}^*, \prec_{\boldsymbol{K}^*})$ is pure. If both $(\boldsymbol{K}_0^k, \prec_k)$ and $(\boldsymbol{K}_0^\ell, \prec_\ell)$ are definable by an $L_{\omega_1,\omega}$-sentence, then the same is true for $(\boldsymbol{K}^*, \prec_{\boldsymbol{K}^*})$.

Proof. Let $\boldsymbol{K}^*$ be the AEC defined by the following construction. The construction contains 4 bipartite graphs entangled together. Recall that a bipartite graph is a τ_0 structure and σ_1 is a τ_0 sentence.

a) A_1, A_2, A_3, C_1, C_2 are non-empty and partition the universe.
b) The structures (A_1, A_2, C_1), (A_1, A_3, C_1), (A_1, C_2, C_1), and (A_2, A_3, C_2) are colored bipartite graphs satisfying σ_1.
c) C_1 is a model in $\boldsymbol{K}_0^k$ and C_2 is a model in $\boldsymbol{K}_0^\ell$. In particular $|C_1| \leq \kappa$ and $|C_2| \leq \lambda$

Define for $\mathcal{M}, \mathcal{N} \in \boldsymbol{K}^*$, $\mathcal{M} \prec_{\boldsymbol{K}^*} \mathcal{N}$, if $\mathcal{M} \subset \mathcal{N}$ with respect[5] to τ_0, $C_1^{\mathcal{M}} \prec_k C_1^{\mathcal{N}}$ and $C_2^{\mathcal{M}} \prec_\ell C_2^{\mathcal{N}}$.

(1) Fix $\chi < \kappa$ and let $\mathcal{M}_1 = (A_1^1, A_2^1, A_3^1, C_1^1, C_2^1)$, $\mathcal{M}_2 = (A_1^2, A_2^2, A_3^2, C_1^2, C_2^2)$ be two models in $\boldsymbol{K}^*$ such that both M_1, M_2 have size $\leq \chi$.

[4]By the union of AEC's with disjoint vocabularies we mean the collection of structures in the union of the vocabularies, where the obvious symbols have the empty interpretation, and one model is a strong substructure of another if the same is true for their reducts to the vocabulary where the structures are non-trivial.

[5]We abuse notation here; depending on the exact location the colored graph will be with respect to a different ternary relation; but we will think of it as a structure modeling the appropriate translation of σ_1.

Use JEP($<\kappa$) on $\boldsymbol{K}_0^\ell$ to extend the τ_ℓ-structures C_2^1, C_2^2 to a common structure $\check{C}_2$ which has cardinality χ. Use the argument of Theorem 3.3.3.1 to find a common embedding of (A_2^1, A_3^1, C_2^1) and (A_2^2, A_3^2, C_2^2) with domain $(A_2^1 \cup A_2^2, A_3^1 \cup A_3^2, \check{C}_2)$ with cardinality χ. Note that the proof of Theorem 3.3.3.1 does not add any vertices to the graph. Then use JEP($<\kappa$) in $\boldsymbol{K}_0^k$ to find a common extension $\check{C}_1$ of C_1^1 and C_1^2 of cardinality χ. Now consider the structures (A_1^1, A_2^1, C_1^1), (A_1^1, A_3^1, C_1^1), $(A_1^1, \check{C}_2, C_1^1)$ and (A_1^2, A_2^2, C_1^2), (A_1^2, A_3^2, C_1^2), $(A_1^2, \check{C}_2, C_1^2)$. Apply the argument of Theorem 3.3.3.1 again several times to find an $\boldsymbol{K}^*$ extension of all these models with domain $(A_1^1 \cup A_1^2, A_2^1 \cup A_2^2, A_3^1 \cup A_3^2, \check{C}_1, \check{C}_2)$. Exactly as in Theorem 3.3.3.1 we verify this structure is in $\boldsymbol{K}^*$.

(2)The proof for AP follows as in Theorem 3.3.3.

(3) and (4) Assume that C_1 has size κ. By Lemma 2.3, if A_1 has size κ^+, then A_2, A_3, C_2 have size $\leq \kappa^+$ and by Theorem 3.4.7 there are 2^{κ^+} many non-isomorphic maximal models in κ^+. If A_1 has size $> \kappa^+$, then A_2, A_3, C_2 have size $\leq \kappa$, and notice that the size of A_1 can be arbitrarily large. If A_1 has size κ, then the sizes of A_2, A_3, C_2 can be greater than κ^+.

Repeating the same argument, assume that C_1 and A_1 have size κ, and C_2 has size λ. If A_2 has size λ^+, then A_3 has size $\leq \lambda^+$ and by Theorem 3.4.7 again, there are 2^{λ^+} many nonisomorphic maximal models in λ^+. If A_2 (or A_3) has size λ, then A_3 (respectively A_2) can have any size and we get arbitrarily large models.

The failure of JEP in $\lambda \geq \kappa$ now fails as Theorem 3.3.3.

(5) The argument is similar to the proof of Lemma 3.3.2. Let M be a model in $\boldsymbol{K}^*$ and X be a subset of M. Find some $\check{C}_1 \in \boldsymbol{K}_0^k$ such that $X \cap C_1^M \subset \check{C}_1$ and $|\check{C}_1| = |X \cap C_1^M| + LS(\boldsymbol{K}_0^k)$. Then find some $\check{C}_2 \in \boldsymbol{K}_0^\ell$ such that $X \cap C_2^M \subset \check{C}_2$ and $|\check{C}_2| = |X \cap C_2^M| + LS(\boldsymbol{K}_0^\ell)$. Let $M_0 = X \cup \check{C}_1 \cup \check{C}_2$. Then M_0 belongs to $\hat{\boldsymbol{K}}_0$. Indeed, M_0 satisfies σ_1, since any violations of $(*)$ in M_0 would be violations of $(*)$ in M too. Contradiction. Considering that $|M_0| \leq |X| + |\check{C}_1| + |\check{C}_2| \leq |X| + LS(\boldsymbol{K}_0^k) + LS(\boldsymbol{K}_0^\ell) = |X| + \max\{LS(\boldsymbol{K}_0^k), LS(\boldsymbol{K}_0^\ell)\}$, it follows that $LS(\boldsymbol{K}^*) = \max\{LS(\boldsymbol{K}_0^k), LS(\boldsymbol{K}_0^\ell)\}$.

Finally observe that the conjunction of (a)-(c) is expressible by an $L_{\omega_1,\omega}$-sentence if and only if membership in both $\boldsymbol{K}_0^k$ and $\boldsymbol{K}_0^\ell$ is expressible by an $L_{\omega_1,\omega}$-sentence. $\square$

We sketch a minor variant in the argument to extend this to infinitely many cardinals.

Theorem 3.5.2. *Let $\langle \lambda_i : i \leq \alpha \rangle$ be a strictly increasing sequence of cardinals. Assume that for each $i \leq \alpha$, there exists an AEC $(\boldsymbol{K}_0^i, \prec_{\boldsymbol{K}_0^i})$ with models in λ_i but no higher. Then if all $(\boldsymbol{K}_0^i, \prec_{\boldsymbol{K}_0^i})$ satisfy JEP($<\lambda_0$), there is an AEC $(\boldsymbol{K}^*, \prec_{\boldsymbol{K}^*})$ which*

1. *satisfies JEP($<\lambda_0$), while JEP fails for all larger cardinals;*
2. *fails AP in all infinite cardinals;*
3. *there exist $2^{\lambda_i^+}$ many nonisomorphic maximal models in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality;*
4. *has arbitrarily large models; and*
5. *$LS(\boldsymbol{K}^*) = \max\{LS(\boldsymbol{K}_0^i) | i \leq \alpha\}$.*

If all $(\boldsymbol{K}_0^i, \prec_{\boldsymbol{K}_0^i})$ are pure, then $(\boldsymbol{K}^*, \prec_{\boldsymbol{K}^*})$ is pure. Further, if $\alpha < \aleph_1$ and all $(\boldsymbol{K}_0^i, \prec_{\boldsymbol{K}_0^i})$ are definable by an $L_{\omega_1,\omega}$-sentence, then the same is true for $(\boldsymbol{K}^*, \prec_{\boldsymbol{K}^*})$.

Proof. Let $\boldsymbol{K}^*$ be the AEC defined by the following construction.

a) The sets $(A_i | i \leq \alpha)$ and $(C_i | i < \alpha)$ are non-empty and partition the universe.
b) For each i, j with $i < j \leq \alpha$, the triples (A_i, A_j, C_i) and (A_i, C_j, C_i) satisfy σ_1.
c) For each $i \leq \alpha$, C_i is a model in $\boldsymbol{K}_0^i$, which implies that $|C_i| \leq \lambda_i$.

Define for $\mathcal{M},\mathcal{N} \in \boldsymbol{K}^*$, $\mathcal{M} \prec_{\boldsymbol{K}^*} \mathcal{N}$, if $\mathcal{M} \subset \mathcal{N}$ with respect to τ_0 and $C_i^{\mathcal{M}} \prec_{\boldsymbol{K}_0^i} C_i^{\mathcal{N}}$, for all $i \leq \alpha$.

The proof is like the proof of Theorem 3.5.1 with some easy modifications. Observe that if for some i, $|C_i| = \lambda_i$ and $|A_i| = |C_i|^+$, then by Lemma 2.3 all A_j, C_j, $j > i$, are "locked" to have size at most $|A_i|$, and by Theorem 3.4.7 there are $2^{\lambda_i^+}$ many nonisomorphic maximal models in λ_i^+. If $|A_i| = |C_i|$, then the cardinalities of A_j, C_j, $j > i$ can be greater than λ_i^+. We leave the rest of the details to the reader. □

We need some background before getting specific applications of the previous theorem. The next fact follows from Theorem 4.20 of [**?**] for $\aleph_r = \kappa$, noting that joint embedding holds in $\aleph_{r-1}$. Indeed, 2-AP in $\aleph_{r-2}$ implies 2-AP of models with cardinality $\aleph_{r-1}$ over models of cardinality $< \aleph_{r-1}$ (or the empty set). This yields a complete sentence ϕ_r whose class of models is denoted At^r; a similar argument for the incomplete sentence with models $\hat{\boldsymbol{K}}^r$ is in Theorem 4.3 of that paper.

Fact 3.5.3. *Every cardinal $\kappa < \aleph_\omega$ is characterized by a (complete) sentence of $L_{\omega_1,\omega}$ that satisfies JEP($< \kappa$).*

We describe the next example, based on [**?**], in detail since the particular formulation is important.

Example 3.5.4. *Fix some countable ordinal α and let $\{\beta_n | n \in \omega\}$ list the ordinals less than α. Consider the vocabulary τ that contains a binary relation $\in$, a unary function r (for 'rank') and constants $(c_{\beta_n})_{n\in\omega}$. Let ϕ_α be the conjunction of the following:*

- $\forall x,\ x \in c_{\beta_n} \leftrightarrow \bigvee_{\beta_i \in \beta_n} x = c_{\beta_i}$*, for each n;*
- $\forall x,\ \bigvee_{n\in\omega} r(x) = c_{\beta_n}$*;*
- $r(c_{\beta_n}) = c_{\beta_n}$*, for each n;*
- $\forall x, y,\ x \in y \rightarrow r(x) \in r(y)$*; and*
- $\forall x, y,\ (\forall z)((z \in x \leftrightarrow z \in y) \rightarrow x = y)$ *(Extensionality).*

Observe that ϕ_α is an $L_{\omega_1,\omega}(\tau)$-sentence. Let $\boldsymbol{K}_\alpha$ be the collection of all models of ϕ_α. If $M \in \boldsymbol{K}_\alpha$, then M can be embedded into V_α. In particular, $|M| \leq |V_\alpha| = \beth_\alpha$.

Fact 3.5.5. *For each $\alpha < \omega_1$, $(\boldsymbol{K}_\alpha, \subseteq)$ satisfies the following.*

(a) $\boldsymbol{K}_\alpha$ has a unique maximal model in cardinality $\beth_\alpha$, and no larger models;
(b) JEP($\leq \beth_\alpha$) holds; and
(c) LS($\boldsymbol{K}_\alpha$) $= \aleph_0$.

Note that under GCH up to $\aleph_\omega$, Fact 3.5.5 is stronger than Fact 3.5.3 since JEP($< \kappa$) is replaced by JEP($\leq \kappa$).

Corollary 3.5.6. *Here are three applications of Theorem 3.5.2.*

1. *If $< \lambda_i | i \leq \alpha \leq \omega >$ is any increasing sequence of cardinals below $\aleph_\omega$, then there exists an $L_{\omega_1,\omega}$ sentence ψ*
   - *(a) whose models satisfy JEP($< \lambda_0$);*
   - *(b) that fails AP in all infinite cardinals;*
   - *(c) has $2^{\lambda_i^+}$ many nonisomorphic maximal models in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality, while JEP fails for all larger cardinals; and*
   - *(d) has arbitrarily large models.*
2. *If $< \beth_{\alpha_i} | i \leq \gamma < \omega_1 >$ is a strictly increasing sequence, then there exists an $L_{\omega_1,\omega}$ sentence ψ'*
   - *(a) whose models satisfy JEP($\leq \beth_{\alpha_0}$);*

(b) fails AP in all infinite cardinals;

(c) has $2^{\beth^+_{\alpha_i}}$ many nonisomorphic maximal models in $\beth^+_{\alpha_i}$, for all $i \leq \gamma$, but no maximal models in any other cardinality, while JEP fails for all larger cardinals; and

(d) has arbitrarily large models.

(3) *If $< \lambda_i | i \leq \alpha \leq \omega >$ is any countable increasing sequence of cardinals below $\beth_{\omega_1}$ that are characterized by complete $L_{\omega_1,\omega}$ sentences, then there exists an $L_{\omega_1,\omega}$-sentence ψ''*

(a) whose models satisfy JEP($\aleph_0$);

(b) fails AP in all infinite cardinals;

(c) has $2^{\lambda_i^+}$ many nonisomorphic maximal models in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality; and

(d) has arbitrarily large models.

Proof. For 1) use Theorem 3.5.2 and Fact 3.5.3. For 2) use Theorem 3.5.2 and Fact 3.5.5. 3) is easy by Theorem 3.5.2 since every complete sentence satisfies JEP in $\aleph_0$. $\square$

Question 3.5.7. *Is there an $L_{\omega_1,\omega}$-sentence that has maximal models in uncountably many cardinals but arbitrarily large models?*

Department of Mathematics, Statistics, and Computer Science, M/C 249 851 S. Morgan Chicago IL 60607

E-mail address: jbaldwin@uic.edu

E-mail address: kwienmart@gmail.com

4001 W.McNichols, Mathematics Department, University of Detroit Mercy, Detroit, MI 48221, USA

E-mail address: souldaio@udmercy.edu